Scheme of Work

Cambridge International A Level
Further Mathematics

9231 Paper 1
For examination from 2017–2019

Cambridge Advanced

Contents

Introduction

This scheme of work has been designed to support you in your teaching and lesson planning. It is important to have a scheme of work in place in order for you to ensure that the syllabus is covered fully and this scheme can be adapted to suit your institution and the levels of ability and learning preferences of your learners.

Timings for activities and feedback are left to the judgment of the teacher, according to the level of the learners and size of the class. Length of time allocated to a task is another possible area for differentiation.

This syllabus contains topics at a high mathematical level, but challenging problems are referenced for learners who require them.

Recommended prior knowledge
Knowledge of the syllabus for Pure Mathematics (units Pure 1 and Pure 3 for syllabus 9709 is required for this syllabus. Some topics can be taught alongside the 9709 course so learners do not have to complete the 9709 course before beginning to study Further Mathematics. These topics are mainly listed early in the scheme of work, but the order of teaching can be varied to suit the order of the 9709 course.

Guided learning hours
Guided learning hours give an indication of the amount of contact time teachers need to have with learners to deliver a particular course. Our syllabuses are designed around 360 hours for Cambridge International A Level. The number of hours may vary depending on local practice and your learners' previous experience of the subject. The table below give some guidance about how many hours are recommended for each topic.

Syllabus ref	Suggested teaching time	Suggested teaching order
1. **Polynomials and rational functions**	It is recommended that this unit should take about 10 hours	**3**
2. **Polar coordinates**	It is recommended that this unit should take about 15 hours	**4**
3. **Summation of series**	It is recommended that this unit should take about 10 hours	**2**
4. **Mathematical induction**	It is recommended that this unit should take about 15 hours	**1**
5. **Differentiation and integration**	It is recommended that this unit should take about 25 hours	**7**
6. **Differential equations**	It is recommended that this unit should take about 20 hours	**9**
7. **Complex numbers**	It is recommended that this unit should take about 20 hours	**5**
8. **Vectors**	It is recommended that this unit should take about 20 hours	**6**
9. **Matrices and linear spaces**	It is recommended that this unit should take about 25 hours	**8**

Resources
The resource list for this syllabus is listed at **www.cie.org.uk.** There are no endorsed textbooks for this syllabus at present, but the list includes useful textbooks which cover much of the syllabus as well as other references.

Teacher support
Teacher Support **https://teachers.cie.org.uk** is a secure online resource bank and community forum for Cambridge teachers, where you can download specimen and past question papers, mark schemes and other resources. We also offer online and face-to-face training; details of forthcoming training opportunities are posted online. This scheme of work is available as PDF and an editable version in Microsoft Word format; both are available on Teacher Support at **https://teachers.cie.org.uk** If you are unable to use Microsoft Word you can download Open Office free of charge from **www.openoffice.org**.

Websites
This scheme of work includes website links providing direct access to internet resources. Cambridge International Examinations is not responsible for the accuracy or content of information contained in these sites. The inclusion of a link to an external website should not be understood to be an endorsement of that website or the site's owners (or their products/services).

Teachers can register free for the STEM website but subsequent to the production of this Scheme of Work the Integral websites is now only available via paid subscription.

The TARSIA software which allows you to create puzzles to test manipulation in a different way to a textbook exercise is free to download from www.mmlsoft.com/index.php/products/tarsia

The website pages referenced in this scheme of work were selected when the scheme of work was produced. Other aspects of the sites were not checked and only the particular resources are recommended.

How to get the most out of this scheme of work – integrating syllabus content, skills and teaching strategies

We have written this scheme of work for the Cambridge International A Level Further Mathematics 9231 Paper 1 syllabus to provide ideas and suggestions of how to cover the content of the syllabus. We have occasionally referred to materials originally written for Pure Mathematics modules of other Further Mathematics syllabuses, frequently written as Further Pure 1, 2, 3 and 4 or shortened to FP1, 2, 3 and 4 and we have indicated in which module they may be found. We have designed the following features to help guide you through your course.

Learning objectives help your learners by making it clear the knowledge they are trying to build. Pass these on to your learners by expressing them as 'We are learning to / about…'.

Suggested teaching activities give you lots of ideas about how you can present learners with new information without teacher talk or videos. Try more active methods which get your learners motivated and practising new skills.

Extension activities provide your more able learners with further challenge beyond the basic content of the course. Innovation and independent learning are the basis of these activities.

Independent study (I) gives your learners the opportunity to develop their own ideas and understanding with direct input from you.

Syllabus ref	Learning objectives	Suggested teaching activities
1. Polynomials and rational functions	Recall and use the relations between the roots and coefficients of polynomial equations, for equations of degree 2, 3, 4 only	**Prior knowledge:** Learners will be able to solve quadratic equations by factorising to identify integer roots **Main theme**: Learners can investigate the connection between the roots of a quadratic equation and the coefficients, and then justify their findings by multiplying out the linear factors and matching coefficients. It's easier to start with the coefficient of x^2 equal to 1. This can then be extended to other coefficients, and to cubics and quartics. The simple identity $$\sum \alpha^2 \equiv \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$ is worth teaching, but the more complicated formulae are sometimes misquoted, so it's probably better to use the equations themselves for higher powers. The Integral website http://integralmaths.org has some notes and examples for polynomials up to degree 3. Look at MEI FP1 which goes up to quartics. or OCR Further Pure 1 which could be used for independent study **(I)** or revision up to cubics **(F)**. There are also notes, with short exercises **(F),** on this topic on the Community Resources area of Teacher Support **https://teachers.cie.org.uk** **Extension activity:** Learners can investigate the relations for polynomials beyond a quartic and find a general rule.

Past examination papers, Specimen Papers and **Mark Schemes** are available for you to download at

https://teachers.cie.org.uk

Using these resources with your learners allows you to check their progress and give them confidence and understanding.

Formative assessment (F) is on-going assessment which informs you about the progress of your learners. Don't forget to leave time to review what your learner has learnt, you could try question and answer, tests, quizzes, 'mind maps', or 'concept maps'. These kinds of activities can be found in the scheme of work.

Mathematical induction

Syllabus ref	Learning objectives	Suggested teaching activities
4. Mathematical induction	Use the method of mathematical induction to establish a given result; Recognise situations where conjecture based on a limited trial followed by inductive proof is a useful strategy, and carry this out in simple cases, e.g. find the *n*th derivative of xe^{x} .	**Prior knowledge**: The level of algebraic handling can be quite high, and learners may need knowledge of geometry for some proofs. **Introduction**: To put this topic in context, you can give some examples of deductive proofs, such as proving the formula for the sum of an algebraic or geometric series, or the quadratic formula, explaining the need to show mathematical rigour at every step. To test these ideas further, some false proofs can be examined, and learners should be encouraged to spot the mistakes in the reasoning. Examples of these can be found on the NRICH website http://nrich.maths.org looking under STEP activities for proof (**I**). Proof by contradiction and by exhaustion could also be examined as an extension activity from the same resource. Some classical proofs are illustrated on undergroundmathematics.org under Divisibility and Induction. **Main theme**: The structure of an induction proof needs careful explanation. The rather simple example on the Khan Academy website www.khanacademy.org does give a good idea of the structure. Look up Proof by Induction. Using the analogy of climbing steps helps – if you can get on the first step AND move from one step to the next, then you can get to the top! Showing that if a proposition is true for some arbitrary value of *n,* say *k,* then it is also true for the case where $n = k + 1$ can be compared to be getting from any step to the next step. Proving the proposition is true for (usually) $n = 1$ is essential to get started on the journey up. The language of an induction proof is very important and learners should practise using it in full, avoiding use of statements like 'assume $n = k$' as short cuts. There is a good matching activity to be found on the STEM website www.stem.org.uk – search for 'Induction Proof' to access the cards. You might start by proving familiar results such as the sum of the positive integers. Some useful examples and exercises can be found on the Integral website http://integralmaths.org where excellent teaching and learning resources are to be found for various Further Mathematics topics. Proof is a Further Pure 1 (FP1) topic on most syllabuses. Divisibility tests, inequalities, calculus, geometry and series may all be used as contexts and it's always important that the deductive step is written out fully with a rigorous argument.

Past examination papers
Past examination papers and mark schemes are available to download at **https://teachers.cie.org.uk** Jun 16 Paper 11 Q3 Divisibility Jun 16 Paper 13 Q2 Geometric property Jun 15 Paper 11 Q3 Inequality Jun 15 Paper 13 Q3 Series Nov 15 Paper 11 Q3 Calculus Jun 14 Paper 11 Q3 Divisibility Jun 14 Paper 13 Q3 Divisibility

Summation of series

Syllabus ref	Learning objectives	Suggested teaching activities
3. Summation of series	Use the standard results for Σr, Σr^2, Σr^3 to find related sums;	**Prior knowledge:** Learners will have done some work on geometric and arithmetic series, and be familiar with the sigma notation. **Introduction:** If induction proof has already been covered, it makes a good start to this topic to prove the three formulae that are going to be used. **Main theme**: These types of problems depend on algebraic manipulation to reduce complex expressions to one or more of the standard formulae, and learners can often find a way to approach a summation themselves, making this a possible learner-led activity. Discussion of different methods followed by a summary of successful approaches should cover all the necessary types. Setting a variety of summations needing the expansion of brackets, subtraction of $\sum_{r=1}^{n} u_r$ from $\sum_{r=1}^{2n} u_r$ and manipulation of odd and even terms prepares learners for the range of techniques needed. Although learners may remember the three formulae, it's a good idea for them to check on the MF10 formulae list given in the syllabus, rather than rely on memory in an examination. There are worksheets on the Integral site http://integralmaths.org which give extra practice at different levels, for example in the OCR FP1 section. **Extension activity:** There are some interesting and challenging examples on http://nrich.maths.org – search for summation of series and filter to get problems at the highest level.

Syllabus ref	Learning objectives	Suggested teaching activities
3. Summation of series	Use the method of differences to obtain the sum of a finite series, e.g. by expressing the general term in partial fractions;	**Prior knowledge:** It is worth revising the process of splitting an algebraic fraction into partial fractions before embarking on this topic, as it is core to many of the problems. Mr Barton's Maths www.mrbartonmaths.com/teachers.htm has some Tarsia domino and jigsaw type puzzles that practise these in a slightly different way to a textbook exercise. You need to download the software first. **Main theme**: Learners need to be shown how to set out their work so that any cancelling patterns are easy to spot, as well as being encouraged to write down at least three or four terms from each end of the summation, to fully establish which terms will remain after cancelling. Detailed working is needed if candidates are asked to show that an expression can be written in partial fractions. Some questions require learners to find the nth term of a sequence by using the difference between successive sums. There are examples and exercises on the Integral site (**F**), as above and on http://mathsathawthorn.pbworks.com – look under FP1.
3. Summation of series	Recognise, by direct consideration of a sum to n terms, when a series is convergent, and find the sum to infinity in such cases.	**Prior knowledge**: Learners will have come across the sum of an infinite geometric progression. **Main theme:** The technique studied here is to investigate the sum of n terms, which is then used to determine whether a series is convergent. It is also used to find the sum to infinity if it exists, so the sum of a GP makes a useful example. Learners might find the videos on www.furthermaths.org.uk helpful. The video from OCR revision on Summation of Series covers this topic as well as the previous elements on series (**I**). mathcentre.ac.uk has comprehensive notes on this too – search for Sum of an Infinite Series. **Extension activity:** This is a good place to investigate the formulae $\sum \dfrac{1}{(k-1)!}$ and get learners to investigate adding more and more terms and see where they get. The series for e and π are also explained on the Mathcentre www.mathcentre.ac.uk website, though some of their material uses a different notation. There are plenty of interesting series to investigate on http://nrich.maths.org – learners might try Telescoping Series or Reciprocal Triangles.

Past examination papers
Past examination papers and mark schemes are available to download at **https://teachers.cie.org.uk** Jun 16 Paper 11 Q2 Difference method with partial fractions Jun 16 Paper 13 Q1 Difference method with partial fractions Jun 15 Paper 11 Q1 – using standard results Jun 15 Paper 13 Q4 Difference method Nov 15 Paper 11 Q4 Difference method Jun 14 Paper 11 Q2 Difference method Jun 14 Paper 13 Q2 Sum to infinity

Polynomials and rational functions

Syllabus ref	Learning objectives	Suggested teaching activities
1. Polynomials and rational functions	Recall and use the relations between the roots and coefficients of polynomial equations, for equations of degree 2, 3, 4 only;	**Prior knowledge:** Learners will be able to solve quadratic equations by factorising to identify integer roots **Main theme**: Learners can investigate the connection between the roots of a quadratic equation and the coefficients, and then justify their findings by multiplying out the linear factors and matching coefficients. It's easier to start with the coefficient of x^2 equal to 1. This can then be extended to other coefficients, and to cubics and quartics. The simple identity $\sum\alpha^2 \equiv (\sum\alpha)^2 - 2\sum\alpha\beta$ is worth teaching, but the more complicated formulae are sometimes misquoted, so it's probably better to use the equations themselves for higher powers. The Integral website http://integralmaths.org has some notes and examples for polynomials up to degree 3. Look at MEI FP1 which goes up to quartics. or OCR Further Pure 1 which could be used for independent study (**I**) or revision up to cubics (**F**). There are also notes, with short exercises (**F**), on this topic on the Community Resources area of Teacher Support **https://teachers.cie.org.uk** **Extension activity:** Learners can investigate the relations for polynomials beyond a quartic, and find a general rule.
1. Polynomials and rational functions	Use a given simple substitution to obtain an equation whose roots are related in a simple way to those of the original equation;	**Main theme:** This follows on from the previous section and PowerPoints supporting the process of substitution are on http://integralmaths.org. If a substitution is given, then using it will be the quickest way to the answer. If no substitution is offered, then the methods of the previous section, using the original equation and manipulating the sums and products of the roots should be relatively straightforward. There are notes, with short exercises (**F**), on this topic on the Community Resources area of Teacher Support **https://teachers.cie.org.uk**.

Syllabus ref	Learning objectives	Suggested teaching activities
1. Polynomials and rational functions	Sketch graphs of simple rational functions, including the determination of oblique asymptotes, in cases where the degree of the numerator and the denominator are at most 2 (detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as turning points, asymptotes and intersections with the axes).	**Prior knowledge**: As well as skills of differentiation, learners need to be able to divide polynomials or find quotient and remainder by equating coefficients. It is worth practising this – using a card matching activity from Mr Barton's wide collection makes for a different type of practice. You need to down load the Tarsia free software first www.mmlsoft.com/index.php/products/tarsia then look at www.mrbartonmaths.com. Algebraic division is under the Core 4 section. Notes on polynomial division can also be found in the Algebra section of the Mathtutor - www.mathtutor.ac.uk along with practice exercises (**F**). **Main theme**: Although learners should be able to sketch without a graphing package, it is a good idea to use either a graphic calculator or a graphing package to inform and confirm their efforts in the early stages. Indeed, it is possible to explore graphs and identify the equations of the vertical asymptotes and then come back and look at the algebra. An open ended task on working out the equation for a given graph can be found on undergroundmathematics.org – look under Polynomials and Functions for 'Can you find – asymptote edition'. The polynomial division needs to be carried through completely, so that the remainder is a proper algebraic fraction otherwise the oblique asymptote will be wrong. The key features of the graphs need to be clearly labelled, especially the coordinates of the intercepts with the axes, the turning points and the trends as x and y tend to infinity. These features are not always accurate enough from graphing packages. Learners also need to find the range of a function which is restricted – this is usually done by rearranging the equation into a quadratic and finding where it doesn't have real roots. A YouTube video called 'Determining Range in Real Rational functions' goes through these steps www.youtube.com/watch?v=cqUW25bzOro (**I**). There are also examples in the section called Find the range of rational functions on www.analyzemath.com.

Past examination papers

Past examination papers and mark schemes are available to download at **https://teachers.cie.org.uk**
Jun 16 Paper 11 Q1 Related cubic Q7 Asymptote, range, sketch
Jun 16 Paper 13 Q1 Roots, Q5 Gradient, asymptotes, sketch
Jun 15 Paper 11 Q4 Related cubic
Jun 15 Paper 13 Question1 Roots Q 10 Range, asymptote, sketch
Nov 15 Paper 11 Q5 Roots Q8 Turning points, sketch
Jun 14 Paper 11 Q1 Roots Q11 OR Asymptotes, range
Jun 14 Paper 13 Q11 EITHER Turning points, sketch

Polar coordinates

Syllabus ref	Learning objectives	Suggested teaching activities
2. Polar coordinates	Understand the relations between cartesian and polar coordinates (using the convention $r \geqslant 0$), and convert equations of curves from cartesian to polar form and *vice versa*;	**Prior knowledge:** learners will be familiar with cartesian coordinates and using radians for angle measure. **Main theme:** A discussion about what is needed to communicate where a point is on a plane can lead into the method of identifying position by giving an angle measured from the initial line, and a distance from the pole. Learners should be able to work out how to convert between the two with some supported investigation. It's important to emphasise the convention that $r \geqslant 0$ is used on this syllabus – this is not always the case with online resources and older textbooks. Lots of practice is needed with the conversions, emphasising the use of $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$. There are notes on the whole topic in the Community Resources area and under Plane Analytic Geometry on www.intmath.com.
2. Polar coordinates	Sketch simple polar curves, for $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$ or a subset of either of these intervals (detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as symmetry, the form of the curve at the pole and least/greatest values of *r*);	**Main theme:** Whilst learners need to be able to plot curves by hand, they can relatively quickly gain valuable familiarity of a range of polar curves by investigating them on a graphic calculator, or using a graphing package. It's very helpful if they recognise some of the common graphs, but they shouldn't try to learn them all and need to be prepared for the unfamiliar. Drawing up a table of values and then making sure that key points are clearly marked on the sketch graphs is important. If learners look for symmetries in trigonometric functions particularly, they can save valuable time in the plotting. GeoGebra is a free graphing package and lots of examples are available for learners to explore on applets – search on www.geogebra.org for polar graphs (**I**). Another site with plenty of sketches is www.intmath.com – look for Curves in Polar Coordinates, under Plane Analytical Geometry. Their examples don't follow the convention that $r \geqslant 0$, so learners need to work out which part of the curve would be lost.

Syllabus ref	Learning objectives	Suggested teaching activities
2. Polar coordinates	Recall the formula $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \mathrm{d}\theta$ for the area of a sector, and use this formula in simple cases.	**Main theme:** This formula can be derived by looking at the area bounded by the curve and two radii at θ and $\theta + \delta\theta$, and finding an expression for $\dfrac{\mathrm{d}A}{\mathrm{d}\theta}$. The formula does need to be learned! The integrations often involve trigonometric expressions, so it's good to go over the integration of e.g. $\cos^2\theta$ and $\sin^2\theta$, but also make learners aware that the integration may require Further Mathematics techniques, such as reduction formulae. There are notes on the whole topic in the Teacher Support Community Resources area **https://teachers.cie.org.uk**. The topic is covered in MEI FP2 **(F)** on the Integral website integralmaths.org as well as on mathsathawthorn.pbworks.com. More notes and examples **(F)** can be found at 17calculus.com

Past examination papers
Past examination papers and mark schemes are available to download at **https://teachers.cie.org.uk**   Jun 16 Paper 11 Q 4 Polar-cartesian, sketch and area   Jun 16 Paper 13 Q7 Polar-cartesian, sketch and area   Jun 15 Paper 11 Q5 Intersection, sketch and enclosed area   Jun 15 Paper 13 Q3 Arc length   Nov 15 Paper 11 Q11 OR Sketch, area and arc length   Jun 14 Paper 11 Q5 Sketch and area   Jun 14 Paper 13 Q4 Cartesian-Polar, sketch and area

Complex numbers

Syllabus ref	Learning objectives	Suggested teaching activities
7. Complex numbers	Understand de Moivre's theorem, for a positive integral exponent, in terms of the geometrical effect of multiplication of complex numbers;	**Prior knowledge:** a reminder of the different ways of writing complex numbers can be given using a Tarsia activity from Mr Barton www.mrbartonmaths.com – look in the Further Maths area, and select 'Complex numbers, all forms'. **Introduction:** De Moivre's Theorem for positive integral components can be investigated by drawing some simple examples and exploring what geometrical transformation has taken place after multiplication. **Main theme:** Learners can work from their basic knowledge of complex number multiplication and see how de Moivre's Theorem works for positive integral components. The notes on mathsupport.mas.ncl.ac.uk are informative.
7. Complex numbers	Prove de Moivre's theorem for a positive integral exponent;	**Main theme: (I)** This proof can be left to learners to attempt, either using the polar or exponential form for z, or by induction, if they've already covered this part of the syllabus. **(I)** **Extension activity:** Many learners will be able to work out proofs for both negative and fractional exponents using the positive integral exponent as a starting point.
7. Complex numbers	Use de Moivre's theorem for positive integral exponent to express trigonometrical ratios of multiple angles in terms of powers of trigonometrical ratios of the fundamental angle;	**Main theme:** It is easy for learners to mix up the methods for this and the next topic. They need to remember that to express multiple angles in terms of the powers of trigonometrical ratios, it is quicker to expand $(\cos\theta + i\sin\theta)^n$ and pick out the real or imaginary part as required.

Syllabus ref	Learning objectives	Suggested teaching activities
7. Complex numbers	Use de Moivre's theorem, for a positive or negative rational exponent – in expressing powers of $\sin\theta$ and $\cos\theta$ in terms of multiple angles, – in the summation of series, – in finding and using the nth roots of unity.	**Main theme:** When learners are asked to **express powers of sin and cos in terms of multiple angles**, they should be encouraged to use the fact that $2\cos\theta = z + \dfrac{1}{z}$ and $2i\sin\theta = z - \dfrac{1}{z}$ rather than the expansion in the previous section, as it will require a lot less manipulation. The section on PPLATO www.met.reading.ac.uk/pplato2 on complex numbers covers this section **(F)**, plus the previous one in section 3.3 of the mathematics notes. Finding the nth roots of unity is covered in detail on a video from those posted by the 'ukmathsteacher', which can be found on www.youtube.com.

Past examination papers

Past examination papers and mark schemes are available to download at **https://teachers.cie.org.uk**
Jun 16 Paper 11 Q 6 De Moivre, related equation
Jun 16 Paper 13 Q 9 De Moivre, related integration
Jun 15 Paper 11 Q 8 Summation of series
Jun 15 Paper 13 Q6 Summation of series
Nov 15 Paper 11 Q10 De Moivre, related equation
Jun 14 Paper 11 Q7 De Moivre, related equation
Jun 14 Paper 13 Q5 Summation of series

Vectors

Syllabus ref	Learning objectives	Suggested teaching activities
8. Vectors	Use the equation of a plane in any of the forms $ax + by + cz = d$ or $\mathbf{r.n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, and convert equations of planes from one form to another as necessary in solving problems;	**Prior knowledge**: There is an overlap here with the vectors in Pure 3, so learners might be taken through the two syllabuses at the same time. Basic knowledge of vector notation, the scalar product and different forms of straight line equations is assumed. **Main theme:** There are useful notes on this at Just the Maths https://archive.uea.ac.uk/jtm – look for section 8.6, which includes some of the properties in the third section as well **(F)**. Notes and exercises **(F)** are also available on the Integral site integralmaths.org, looking at MEIFP3 or OCR FP3 sections on vectors.
8. Vectors	Recall that the vector product $\mathbf{a} \times \mathbf{b}$ of two vectors can be expressed either as $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector, or in component form as $(a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$;	**Main theme**: The definition and execution of the vector product are very important – there are different ways of carrying it out, but learners need to be confident and accurate with whichever method is chosen. Sign errors are common. There are some videos on the Mathcentre site www.mathcentre.ac.uk , some PowerPoints on the MathedUp! website www.mathedup.co.uk are helpful as are the notes on Just the maths archive.uea.ac.uk/jtm/contents.htm **(F)**.

Syllabus ref	Learning objectives	Suggested teaching activities
8. Vectors	Use equations of lines and planes, together with scalar and vector products where appropriate, to solve problems concerning distances, angles and intersections, including: – determining whether a line lies in a plane, is parallel to a plane or intersects a plane, and finding the point of intersection of a line and a plane when it exists, – finding the perpendicular distance from a point to a plane, – finding the angle between a line and a plane, and the angle between two planes, – finding an equation for the line of intersection of two planes, – calculating the shortest distance between two skew lines, – finding an equation for the common perpendicular to two skew lines.	**Main theme:** This section of the syllabus is problem solving using the properties above, but it's worth going through examples of these standard elements so that learners don't have to go back to first principles every time. The Distance Dominoes and the Line Groupings in the Active Learning section of Integral's MEI FP3 section on vectors integralmaths.org make good practice and are more interesting than book exercises **(F)**. Although it's difficult to sketch well in 3 dimensions a basic representation of the problem to be tackled does often help. Just the Maths archive.uea.ac.uk/jtm section 8.5 and 8.6 go over these techniques, and has some examples **(F)**. It is also worth looking at the Integral website, integralmaths.org for notes and questions **(F)** – vector methods are covered in OCR Further Pure 3 for example. Although practising the individual methods is helpful, because of the problem-solving nature of this module, it's worth putting together worksheets of past examination questions to give learners the chance to select appropriate methods **(F)**. Some learners will prefer to learn formulae for individual techniques, others will prefer to go back to basics each time – the back to basics approach is very robust! None of the standard formulae are on the MF10 formulae list. **Extension activity**: Challenging problems can be found at NRICH http://nrich.maths.org – the activity called Walls is an open ended investigation of perpendicular planes, Tetra Perp looks into perpendicular edges for a tetrahedron. This website also lists some relevant STEP problems.

Past examination papers

Past examination papers and mark schemes are available to download at **https://teachers.cie.org.uk**
Jun 16 Paper 11 Q 8 Planes, angle and intersection
Jun 16 Paper 13 Q11 OR Distance between lines, planes.
Jun 15 Paper 11 Q11 OR Area, volume
Jun 15 Paper 13 Q8 Perpendiculars, lines
Nov 15 Paper 11 Q11 EITHER Lengths, angles
Jun 14 Paper 11 Q11 Length, planes
Jun 14 Paper 13 Q11 OR Plane, perpendicular, distance

Differentiation and integration

Syllabus ref	Learning objectives	Suggested teaching activities
5. Differentiation and integration	Obtain an expression for $\dfrac{d^2 y}{dx^2}$ in cases where the relation between y and x is defined implicitly or parametrically;	**Prior knowledge:** learners need to have the full range of calculus skills up to Pure 3 level, including parametric and implicit differentiation. **Main theme:** This is relatively straightforward application of earlier calculus skills, which requires a lot of stamina and precision sometimes. Learners need to practice these skills extensively to make sure they can handle the notation correctly, and avoid common errors with the second derivatives in particular. Some notes on second derivatives of parametric equations can be found on the Mathscentre website and examples on implicit functions can be found on the Interactive Mathematics Site in section 9 of the differentiation section www.intmath.com. Some exercises **(F)** can be found on 17calculus.com
5. Differentiation and integration	Derive and use reduction formulae for the evaluation of definite integrals in simple cases;	**Prior knowledge:** a quick revision of integration by parts from Pure 3 will set learners up for this topic. **Main activity:** A simple starting point such as $I = \int e^x \sin(x)\,dx$ where I re-appears on the right hand side gives the idea for a reduction formula before embarking on a more general case. Reduction formulae often require the integrand to be split to create a multiple of the original I. These splits may be algebraic (splitting $f(x)^{\frac{3}{2}}$ in to $f(x) \times f(x)^{\frac{1}{2}}$) or trigonometric (e.g. re-writing $\cos^2 x$ as $1 - \sin^2 x$). Examination questions often start with a differentiation, and using the derivative will inform the choice of integrand. See the 2016 questions for examples. There are two collections of past exam questions **(F)** on the A Level section of Mr Barton's Maths website – under FP2 www.**mrbartonmaths**.com and on the Mathsbank site **(F)** www.mathsbank.co.uk/home/a-level also under FP2.

Syllabus ref	Learning objectives	Suggested teaching activities
5. Differentiation and integration	Use integration to find: – mean values and centroids of two- and three dimensional figures (where equations are expressed in cartesian coordinates, including the use of a parameter), using strips, discs or shells as appropriate, – arc lengths (for curves with equations in cartesian coordinates, including the use of a parameter, or in polar coordinates), – surface areas of revolution about one of the axes (for curves with equations in cartesian coordinates, including the use of a parameter, but not for curves with equations in polar coordinates).	**Prior knowledge**: Examination questions may involve the whole range of calculus techniques, including reduction formulae. **Main theme**: There is a lot of learning to be done here, as learners need to memorise all the formulae. Although they will not need to prove them, it's a good idea to go through step-by-step proofs, so that learners have a better chance to work out the formulae if their memory lets them down in an examination. There are notes on these topics in the Community Resources on Teacher Support **https://teachers.cie.org.uk** as they are not often covered in textbooks or other A level syllabuses. Notes and examples of some applications can also be found in the Just the Maths booklets on archive.uea.ac.uk/jtm (**F**). **Extension activity:** Many STEP problems involve advanced integration techniques, and provide useful challenges to learners. They can be found in various locations, including the Cambridge admissions website www.admissionstestingservice.org

Past examination papers

Past examination papers and mark schemes are available to download at **https://teachers.cie.org.uk**
Jun 16 Paper 11 Q5 Reduction formula, Q11EITHER Arc length, Surface area
Jun 16 Paper 13 Q4 Arc length, Q6 Reduction formula
Jun 15 Paper 11 Q7 Reduction formula, Q9 Arc Length, Mean value
Jun 15 Paper 13 Q5 Reduction formula
Nov 15 Paper 11 Q9 Reduction Formula
Jun 14 Paper 11 Q7 Reduction Formula
Jun 14 Paper 13 Q8 Arc length, surface area Q10 Reduction formula

Matrices

Syllabus ref	Learning objectives	Suggested teaching activities
9. Matrices and linear spaces	Evaluate the determinant of a square matrix and find the inverse of a non-singular matrix (2×2 and 3×3 matrices only), and recall that the columns (or rows) of a square matrix are independent if and only if the determinant is non-zero;	**Prior knowledge:** some learners may have studied basic matrix operations prior to starting on the Further Mathematics course, but if not, they will need to run through the basics. Notes and exercises can be found from Just the Maths archive.uea.ac.uk/jtm from section 9 onwards. **Extension:** the properties of matrices as transformations are tested fully by the problem Nine Eigen on the NRICH website (Extension) http://nrich.maths.org **Main theme:** The process of finding an inverse does need to be learned and performed with accuracy – whilst it is fine to check the answer with a calculator, full working is needed. There are explanations, examples and exercises on the Integral website integralmaths.org, the topic is covered in MEI FP2 or OCR FP1 and some notes (**I**) on the MathedUp! website under FP4.
9. Matrices and linear spaces	Understand the terms 'eigenvalue' and 'eigenvector', as applied to square matrices; Find eigenvalues and eigenvectors of 2×2 and 3×3 matrices (restricted to cases where the eigenvalues are real and distinct);	**Main theme**: there are various techniques for finding the eigenvectors, but it is important that learners take great care when using the vector product method for the eigenvectors not to make errors. Some self-checking can be done, as eigenvalues will be distinct and real (and non-zero). The geometric significance of the eigenvalue and eigenvector should be explained. Calculators can be used to check answers worked calculated by the chosen method. It's worth emphasising that the eigenvalues of a triangular matrix are the entries on the diagonal. – this can save a lot of time! Notes on finding eigenvalues and eigenvectors on both Just the Maths and Integral as above. There are practice questions there too. There are also some notes on the MathedUp! website www.mathedup.co.uk.
9. Matrices and linear spaces	Express a matrix in the form $\mathbf{QDQ^{-1}}$, where $\mathbf{D}$ is a diagonal matrix of eigenvalues and $\mathbf{Q}$ is a matrix whose columns are eigenvectors, and use this expression, e.g. in calculating powers of matrices.	**Main theme:** This process is relatively straightforward, though care should be taken as the process will only work if the eigenvectors which make up Q are indeed linearly independent (this property is discussed in more depth in the next topic). Care must be taken to make sure that the eigenvalues and vectors are in corresponding positions. There are notes and examples of this method on the MathedUp! website www.mathedup.co.uk, under FP4.

Linear spaces

Syllabus ref	Learning objectives	Suggested teaching activities
9. Matrices and linear spaces	Recall and use the axioms of a linear (vector) space (restricted to spaces of finite dimension over the field of real numbers only); Understand the idea of linear independence, and determine whether a given set of vectors is dependent or independent; Understand the idea of the subspace spanned by a given set of vectors; Recall that a basis for a space is a linearly independent set of vectors that spans the space, and determine a basis in simple cases; Recall that the dimension of a space is the number of vectors in a basis.	**Introduction/Extension activity:** It is good to give some background on different mathematical structures such as sets and groups before embarking on this, and useful to have covered the matrix work first to make the links and be able to use matrix transformations as a key example. Learners may work independently to investigate these structures for themselves **(I)** – the Integral website has notes and examples. **Main theme:** There are a lot of abstract concepts in this module on linear (vector) spaces, and learners will probably take some time to absorb them. The definitions and properties need careful explanation and, wherever possible, the more examples from familiar areas the better. As this is not a common topic at A level, it is useful to have an undergraduate textbook to support teaching and learning. Lipschutz, S and Lipson, M, 2013, *Shaum's Outlines of Linear Algebra*, McGraw Hill (as listed in the resources centre www.cie.org.uk) covers the syllabus in good depth, with lots of helpful examples and exercises **(F)**. It also covers the syllabus for Matrices (and goes beyond). Another comprehensive textbook on this subject, including the work on vectors **(F)**, Linear Algebra by Howard Anton, can be downloaded free from Elementary linear algebra 10th edition - University of Warwick. The resource centre www.cie.org.uk also provides several sets of notes.

Syllabus ref	Learning objectives	Suggested teaching activities
9. Matrices and linear spaces	Understand the use of matrices to represent linear transformations from $R^n \rightarrow R^m$. Understand the terms 'column space', 'row space', 'range space' and 'null space', and determine the dimensions of, and bases for, these spaces in simple cases; Determine the rank of a square matrix, and use (without proof) the relation between the rank, the dimension of the null space and the order of the matrix;	Here the syllabus focuses on matrices which represent linear transformations as an example of a linear space, and there are a lot of technical terms to learn and techniques to practice. Working through examples helps to make the link between the concepts and the reality of matrix transformations, and it is worth compiling a worksheet of past questions in addition to those listed below.
9. Matrices and linear spaces	Use methods associated with matrices and linear spaces in the context of the solution of a set of linear equations.	This final part of the syllabus brings the two parts of this topic together, using the properties of linear spaces to inform the role of matrices in the solution of linear equations.

Past examination papers
Past examination papers and mark schemes are available to download at **https://teachers.cie.org.uk** Jun 16 Paper 11 Q10 Eigenvalues/vectors, diagonalization Q11 OR Rank, basis, multiplication Jun 16 Paper 13 Q3 System of equations Q11 EITHER Eigenvalues/vectors, diagonalization Jun 15 Paper 11 Q2 System of equations Q10 Eigenvalues/vectors, diagonalisation Jun 15 Paper 13 Q11 EITHER Vector space OR Eigenvalues/vectors Nov 15 Paper 11 Q6 Eigenvalues/vectors, diagonalisation Q7 Vector space Jun 14 Paper 11 Q6 Vector space Q9 Eigenvalues/vectors Jun 14 Paper 13 Q1 Vector space Q8 Eigenvalues/vectors, diagonalisation

Differential equations

Syllabus ref	Learning objectives	Suggested teaching activities
6. Differential equations	Recall the meaning of the terms 'complementary function' and 'particular integral' in the context of linear differential equations, and recall that the general solution is the sum of the complementary function and a particular integral; Find the complementary function for a second order linear differential equation with constant coefficients; Recall the form of, and find, a particular integral for a second order linear differential equation in the cases where a polynomial or e^{bx} or $a \cos px + b \sin px$ is a suitable form, and in other simple cases find the appropriate coefficient(s) given a suitable form of particular integral;	**Prior knowledge:** learners will have come across simple differential equations of the first order in the Cambridge International AS & A Level Mathematics course. **Main theme**: The language here is important so that learners remember the correct terms for the different parts of the solution. Learners need to learn the complementary functions relevant to each of the three possible types of solution to the auxiliary equation, and to know how to choose a suitable form for the particular integral in the three cases required. For independent study **(I)** learners might find the three videos on the Mathsfaculty site themathsfaculty.org helpful as they go through the three possibilities for the complementary function. This explains carefully how the different forms arise. Questions should not always involve x and y, and it's really important that learners use the variables of the questions consistently, not automatically reverting to x and y. Where an appropriate form for the particular integral is given, learners should use it, and not waste time trying to find it for themselves. There are notes, examples and exercises on the PPLATO site www.met.reading.ac.uk/pplato2 , in Module 6.3 and on the Mathscentre site www.mathcentre.ac.uk (search for 2nd order differential equations), as well as the Integral site http://integralmaths.org - the topic is covered in OCR FP3 **(F)**. **Extension activity**: There are some challenging problems on the NRICH site http://nrich.maths.org, which make learners really think about differential equations in context.

Syllabus ref	Learning objectives	Suggested teaching activities
6. Differential equations	Use a substitution to reduce a given differential equation to a second order linear equation with constant coefficients;	Substitutions are given, but having 'extra' variables in a question can be confusing, so learners need to take great care to use the correct variables at each stage. A PowerPoint with a worked example can be found on www.examsolutions.net/maths-revision/further-maths **(I)**. The videos of worked exam questions on the MathedUp! website are helpful for revision too – 2nd order differential equations comes under FP3.
6. Differential equations	Use initial conditions to find a particular solution to a differential equation, and interpret a solution in terms of a problem modelled by a differential equation.	As well as using the boundary conditions to find the particular solution, learners need to be able to identify the limits of the solution and what the solution represents in terms of the problem. Some practice can be found on www.ncl.ac.uk/students/mathsaid/resources/academic , mathsupport.mas.ncl.ac.uk and www.reading.ac.uk/AcaDepts/sp/PPLATO/imp/h-tutorials. **Extension activity**: There are some challenging problems on the NRICH site http://nrich.maths.org, which make learners really think about differential equations in context.

Past examination papers

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Jun 16 Paper 11 Q9
Jun 16 Paper 13 Q10
Jun 15 Paper 11 Q11
Jun 15 Paper 13 Q9
Nov 15 Paper 11 Q2
Jun 14 Paper 11 Q4
Jun 14 Paper 13 Q10

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