FURTHER MATHEMATICS

Paper 9795/01
Further Pure Mathematics

Key messages

There are significant points that candidates need to bear in mind before they sit this paper, many of which are highlighted again in the later comments on individual questions. Firstly, this is a long paper and some students are not able to complete attempts at all questions. Thus, for each individual candidate, it is very important to choose a suitable order in which to attempt the questions, one that maximises the opportunities to gain high marks. Also, the majority of candidates have little time in which to review or correct working, so care must be taken to ensure that work is done without the need for re-evaluation. Usually, candidates are best served by a thoughtful approach to each question that considers – in particular – how the question is structured (where appropriate); many candidates frequently ignore the very careful signposts offered within the question. Similarly, candidates must be aware of how appropriate the working they are writing down is to the demand of that question or question-part; producing up to a page-and-a-half of working for a result that has been assigned just 1 or 2 marks is clearly inappropriate, and candidates should allow themselves to be guided by the number of marks.

Another key point is that many candidates seem unwilling to deploy their 'single maths' skills in the further maths setting and they should be aware that understanding and fluency in GCSE level and Maths Principal level techniques are taken as 'background/assumed knowledge' within the further maths papers.

General comments

Overall, the quality of the candidates' work was impressive, with over 25 per cent of the entry gaining total scores of at least 100 out of the overall total of 120 marks. There is a small amount of evidence that some candidates found the paper too long to complete within the allotted time. It was usually the case that these candidates had, often at several points in the examination, attempted questions by lengthy methods.

One of the most obvious, and widespread, factors that detracted from the overall performance of the candidature as a whole was the apparent unwillingness to address those (usually small) parts of questions that required explanation or descriptions. This is almost always the case with some parts of a 'Groups' question (**Question 8** on this paper), but also applied to the very final parts to **Questions 4**, **9** and **12**. In addition, candidates often did not take enough care about providing necessary details for given answers, of the 'show that' variety … of which there were several, sprinkled throughout the paper (**Questions 1**, **3**, **6**, **8**, **9**, **12** and **13**). Candidates should, as part of their preparation, be made aware that they must take the time and trouble to justify fully how these given results arise, and not think they can simply jump straight to the answer. Marks will be lost if their working is not convincing.

Several questions on the paper had their own, individual 'gradient of difficulty', and only **Question 12** was especially technically demanding. Although **Question 13** looked rather daunting, the bulk of it was actually quite straightforward once one had got going.

Comments on specific questions

Question 1

This was intended to be a straightforward *entrée* into the paper, and it served its purpose very well indeed, with almost all candidates gaining full marks. Those who did not do so usually fell down when the answer appeared without any intermediate factorisation attempt of the preceding quartic (or, with the n taken out first, cubic) expression.

Question 2

This question was also intended to be a quick and simple test of a single result, and most candidates quoted the direct formula for the shortest distance and got on with it. For the most part, marks were lost as a result of missing (or extraneous) negative signs during the calculation of the cross-product. A few candidates used much lengthier methods – such as finding the distance between two relevant parallel planes, or finding the point on each line that was closest to the other line (at the ends of the mutual perpendicular) – and lengthier working naturally gave more opportunity for slip-ups in the working.

Answer: 4

Question 3

This was a relatively straightforward question, although candidates found a variety of ways to lose marks. In many cases, part **(i)** provided one instance of the 'not fully justifying a given answer' issue raised in the opening remarks. Very few candidates indeed approached the non-real roots of the ensuing quadratic in x by using the fact that the discriminant should be negative. Many candidates simply noted that $\Delta \geqslant 0$ (without explaining what case, or cases, this covered) and proceeded to present the given interval without any mention at all of why this was the one required. This led to the loss of 2 of the 4 marks for these candidates.

In part **(ii)**, there were a lot more cases when 2 more marks were lost, although it could have been all 4 of them had the 'non-**deduce**' approach of differentiation not been awarded some credit on this occasion. This had initially been to allow access to some marks for those candidates who had been unable to do part **(i)** at all, but turned out instead to benefit a large number of candidates who ignored the carefully emboldened word '**Deduce**' at the start of the sentence.

Answer: **(ii)** $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{5}{2}, \frac{9}{2}\right)$

Question 4

This was intended to be a straightforward question but turned out to be one of the poorest-scoring of all the questions on the paper. Part **(i)** was usually completed by showing that the determinant of the coefficient matrix was zero, but part **(ii)** was handled very poorly indeed as around half of all candidates did not make any attempt to answer the question that had been asked. Many of these candidates went on to find a unique solution to the system, despite the fact that they had already concluded that one such did not exist. Many candidates simply went on to show that there was no unique solution by algebraic methods; many others managed to demonstrate that there were no solutions at all. This despite the clear implication in the wording of the question that, not only was there a solution, but they should actually be attempting to find it (in some form or other). Many of the lost marks were due to the omission of an attempt to 'solve' the system at all.

Answer: **(ii)** $(x, y, z) = (t, t - 2, 7 - 2t)$, for example.

Question 5

Overall, this question was done exceptionally well, with high marks scored by most candidates. There were, however, several common shortcomings that led to lost marks: incorrectly solved quadratic auxiliary equations; incorrect complementary functions for the case of complex roots; and incorrect forms for the particular integral, mostly when candidates thought it should be of the form $ax.\mathrm{e}^{2x}$.

Answer: $y = \mathrm{e}^{2x}(A \cos x + B \sin x + 24)$

Question 6

This was another generally successful question, although there were, again, a few common points where marks were lost. In part **(i)**, almost all candidates evaluated f(2.5) and f(3) for suitably-chosen functions f, but many omitted to explain why this led to the given answer. Part **(ii)** was almost universally successfully handled, although some candidates ignored the series results in the Formula Book, preferring instead to derive it from scratch using repeated differentiation. Although there is nothing wrong with this approach, it did waste quite a bit of their time unnecessarily. Most, but not all, candidates spotted the intended method in part **(ii)** of solving a quadratic in x^4.

Answer: **(iii)** 2.7698

Question 7

This was intended to be a reasonably standard complex numbers question, and most candidates scored well on part **(i)** by using the modulus-argument approach. Part **(ii)** was not so successfully approached, despite its intended simplicity. The sketch required little more than an equilateral triangle with vertices in approximately the correct (follow through) positions in the complex plane. However, many triangles looked anything but equilateral. Many marks were then lost to candidates who used a range of complicated trigonometrical approaches to finding 'heights' of triangles, when the formula $\frac{1}{2}ab\sin C$ could be applied reasonably routinely.

Answers: **(i)** $(r, \theta) = \left(\sqrt{2},\ \frac{1}{12}\pi\right), \left(\sqrt{2},\ \frac{9}{12}\pi\right)$ and $\left(\sqrt{2},\ \frac{17}{12}\pi\right)$ **(ii)** $\frac{3}{2}\sqrt{3}$.

Question 8

Groups questions have traditionally been found either too easy or too demanding; this one fell nicely in-between, with an easy part **(i)** and a tougher part **(ii)**. As mentioned in the initial remarks, this question gave rise to several places where an explanation, or some clarifying detail, was required … which candidates seemed unclear as to how to supply. For instance, in part **(i)(b)**, candidates were not permitted to say simply 'Closure – see table', but were required to explain *how* the table demonstrated the closure property. Part **(ii)** provided its own set of difficulties, beginning with the need to identify six distinct elements for *H*. Even the group table of part **(ii)(a)** proved difficult for the majority of candidates, with few of them scoring more than 2 of the 4 marks available. In many cases, candidates would switch from calculating the product term in the table via row × column to column × row, and this would often completely undermine everything that followed in part **(ii)**. In part **(ii)(b)**, very few candidates were clear about how to identify subgroups, and the Yes/No vote for isomorphism was approximately equally split.

Answers: **(i)**

G	**1**	**2**	**4**	**8**	**16**	**32**
1	1	2	4	8	16	32
2	2	4	8	16	32	1
4	4	8	16	32	1	2
8	8	16	32	1	2	4
16	16	32	1	2	4	8
32	32	1	2	4	8	16

(ii)(a)

H	e	x	y	y^2	xy	yx
e	e	x	y	y^2	xy	yx
x	x	e	xy	yx	y	y^2
y	y	yx	y^2	e	x	xy
y^2	y^2	xy	e	y	yx	x
xy	xy	y^2	yx	x	e	y
yx	yx	y	x	xy	y^2	e

(b) $\{e, x\}$, $\{e, xy\}$ and $\{e, yx\}$ of order 2; $\{e, y, y^2\}$ of order 3.

Question 9

This was a question on the relationships between roots and coefficients of a polynomial equation, but with an increasing demand as the question progressed. Thus, part **(i)** was a basic 'write down' part; part **(ii)** involved little more than a manipulation of the standard result $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, albeit in two slightly different settings; part **(iii)** was a test of candidates' ability to identify, and collect together, terms of the same kind. Only part **(iv)** required a more thoughtful understanding of mathematical phraseology. Thus, apart from a few sign mix-ups, parts **(i)** and **(ii)** were high-scoring, and part **(iii)** was generally manageable, other than marks that were lost due to the appearance of given final answers without fully supporting prior working. The 1 mark for part **(iv)**, however, was gained by very few candidates. This was due to almost all candidates not recognising the full requirement of the question as a result of, presumably, a lack of grasp of the meaning of the phrase 'if and only if'. The other obstacle was the incorrect interpretation of the phrase 'one root is twice the product of the other two' to mean $\alpha = 2\beta\gamma$ rather than either $\alpha = 2\beta\gamma$ or $\beta = 2\gamma\alpha$ or $\gamma = 2\alpha\beta$. So, although many candidates scored 8 or 9 marks on the question, very few indeed scored 10 or 11.

Answers: **(i)** $\alpha + \beta + \gamma = a$, $\alpha\beta + \beta\gamma + \gamma\alpha = b$, $\alpha\beta\gamma = c$ **(ii)** $a^2 - 2b$ and $b^2 - 2ac$.

Question 10

This was a relatively straightforward Polar Coordinates question. Many candidates lost 2 or 3 marks on the sketch alone, with many sketches missing key features as candidates did not attempt to find the points at which $r = 0$. This usually led to the lower portion of the sketch being completely wrong. In part **(ii)**, the standard work went very smoothly for the most part, although quite a few candidates couldn't produce the correct double-angle formula for $\sin^2\theta$, and there were often incorrect signs that followed in the integrations. Thus, the correct answer was duly found, although not always from correct working.

Answers: **(ii)** $\frac{3}{4}\pi$.

Question 11

On the whole, this was found to be a very easy question and was done well by about three-quarters of the candidature. In some cases, it was not helpful in part **(i)** that candidates could often take a long time to determine the next four terms of such a well-known sequence. A few candidates interpreted the phrase 'write down the values of F_3 to F_6' as 'write down the values of F_3 *and* F_6' without ensuring the intermediate terms were present on the page. In part **(ii)(b)**, those who made the correct conjecture usually gained all 5 marks. Unfortunately, many candidates opted for a formula without any Fibonacci numbers in at all (regardless of the clear directive to this effect within the question) and others had a mix of Fibonacci numbers and linear terms in n that restricted them to a maximum of 2 of the 5 marks available. A small handful of candidates did not notice the need to take $n = 2$ as the base-line case (though those who chose to use $n = 1$ and identified F_0 as 0 still qualified for full marks).

Answers: **(i)** 2, 3, 5 and 8 **(ii)(a)** $\mathrm{p}_2(x) = \frac{x+2}{x+1}$, $\mathrm{p}_3(x) = \frac{2x+3}{x+2}$, $\mathrm{p}_4(x) = \frac{3x+5}{2x+3}$ **(ii)(b)** $\mathrm{p}_n(x) = \frac{F_n\,x + F_{n+1}}{F_{n-1}\,x + F_n}$

Question 12

This was the most technically demanding question on the paper … involving identities of hyperbolic functions, calculus and explanation/interpretation. Most candidates managed to earn at least two marks in part **(i)** along with around three marks in part **(ii)(a)**. All that was required to wrap up the given approximate answer here was to note that e^{-kn} (where $k = 1$, 2 or, sometimes, 3) tends to zero as n tends to infinity, and every correct exponential form for the final integrated answer afforded candidates the opportunity to argue this result appropriately. Part **(ii)(b)** required candidates to note that the curve essentially becomes a straight line – many did so, but a few were not entirely clear why this led to the given answer.

Question 13

The opening part to this question was on a small part of the course previously untested, namely differentiating inverse-trigonometric functions, and thus part **(i)(a)** caused a lot of difficulty, especially with the first requirement. Many candidates mixed inverses with reciprocals and wrote down worthless material. There are other ways to differentiate $\sec^{-1}x$ and, on this occasion, 'non-deduce' methods were allowed, since the initial result was intended to be a means of guiding candidates in the right direction, and the alternative approaches were mathematical.

However, candidates really must be made aware that, if they do not follow clear (and sometimes helpfully emboldened) directives such as 'hence' and 'deduce' within a question, they risk the loss of several marks on a paper as a whole. Whenever suitable, the Examiners will make a judgement as to whether alternative approaches should be given all, some, or none of the marks available, but it is poor examination technique, and a very risky strategy, to ignore the clear guidance supplied within a question. There is no guarantee that an approach accepted one year will be permitted in future years.

The remainder of the question was then more along the lines of a longer single-maths question – apart from the types of function being used, of course – and most candidates who made a good attempt ended up scoring 12+ marks overall on the question.

Answer : **(i)(b)** $x\sec^{-1}x - \cosh^{-1}x\ (+\ c)$ **(ii)(a)** $P = \left(\sqrt{2},\ \tfrac{1}{4}\pi\right)$, $Q = \left(\sqrt{2} - \tfrac{\pi\sqrt{2}}{4},\ 0\right)$

FURTHER MATHEMATICS

Paper 9795/02
Further Applications of Mathematics

Key messages

- Give full details in "show that" questions
- Understanding of concepts of modelling assumptions and conditions, etc., especially in statistics
- Ensure that the answer given matches the question set

General comments

This paper was easier than last year's paper and marks were correspondingly higher, although a number of candidates revealed gaps or misunderstandings. This particularly applies to verbal questions in statistics. Standard processes were generally carried out well, although questions which required candidates to "show that" were usually answered less well, lacking essential detail.

Candidates would smooth the path for themselves in mechanics questions in particular by making their notation clear and explaining where their equations come from. Such good practice encourages clear thinking as well as helping the Examiner to see what they are trying to do.

Comments on specific questions

Section A: Statistics

1 Most candidates found part **(i)** a straightforward start to the paper. Few multiplied by 500/499, but when n is as large as this it makes only a tiny difference and full marks could be obtained without it.

In part **(ii)** many more candidates needed to realise that two different distributions were involved, and indeed what the word "assume" means in this context. A typical answer was "Yes, we can assume that it is normally distributed because n is large", "it" here presumably referring to the population of all journey times. The distribution asked about in the question, the (parent) distribution of individual journey times, either is or is not normal, regardless of any sample that is chosen from it. But the calculation in part **(i)** involved the distribution of a different variable, that of the mean of a sample of size 500, and the Central Limit Theorem applies to the distribution of such sample means. A full answer would be "No, we do not need to assume that journey times are normally distributed, because the Central Limit Theorem tells us that the sample mean journey time *is* approximately normally distributed, for such a large sample." Centres might find it useful to emphasise the difference between the distribution of a single journey time (effectively a sample of size 1, even if drawn from a very large population) and the distribution of the mean of a large sample.

A few candidates insist on thinking that the Central Limit Theorem has something to do with the value of the variance.

Answer: **(i)** (71.5, 78.5)

2 Most candidates found part **(i)** easy, usually finding the probability that 175g or more was needed. Almost everyone knew that they had to consider a combined distribution in part **(ii)**, with only a minority wrongly calculating the variance as $4^2 \times 1200 + 3^2 \times 1500$.

Answers: **(i)** 0.235 **(ii)** 0.267

3 The requirement to derive a given formula for the PGF of a binomial distribution generally called for more care and detail than candidates were willing to give it. Details such as collecting $p^r t^t$ as $(pt)^r$ were needed, as was indication of where the series ends (rather than just …); Σ notation should be encouraged here.

Part **(ii)** was generally done well although some weaker candidates tried to answer a question they had seen before and worked out $G'(1)$ or equated $G'(t)$ to 1.

Answer: **(ii)** 4.07×10^{-5}

4 As noted in **question 1**, candidates need to have a better understanding of the specific nature of the word "assumption" in this type of question. It refers to conditions that are not inherent in the information already given, so that "goals occur randomly", or even "both sides play for the same length of time" are irrelevant. The only further assumption not inherent in the statement that both goals scored by the home team and goals scored by the away team have Poisson distributions is that the scoring of goals by the two sides is independent. (Strangely, although the conditions seem unlikely, in practice it has been found repeatedly that Poisson distributions do give a good model for this scenario.)

Many misread part **(ii)(a)** or tried to use a calculator; the question tested knowledge of the Poisson formula. In part **(b)** quite a few candidates assumed that the probability of a score of 1–1 was the same as the probability of a total score of 2.

Answer: **(ii)(b)** 0.0798

5 Many candidates could not quote the validity conditions for a normal approximation to binomial. Many said "n is large and p is small", which are the conditions for the normal approximation to the Poisson, while others wrote "n is large and $np > 5$", omitting the second condition which is either "p is close to $\frac{1}{2}$" or "$nq > 5$".

The rest of this question was not dissimilar to last year's and was well answered. There were the usual errors with the continuity corrections, and many also forgot that, while their equations in part **(ii)** had given them σ, this had to be squared to give them npq. Most realised that the value of n had to be an integer.

Answers: **(ii)** $\mu = 21.0$, $\sigma = 3.69$ **(ii)** $n = 60$, $p = 0.35$

6 The majority of candidates could derive the given MGF but, as it involves an infinite integral of an exponential it is necessary to write it in a form that clearly converges, or otherwise indicate the issue. Thus $\left[-\dfrac{4e^{x(t-2)}}{(t-2)^2}\right]_0^\infty$ cannot be taken to have a finite upper limit without further comment; it needs to be written as $\left[-\dfrac{4e^{-x(2-t)}}{(t-2)^2}\right]_0^\infty$, or some other indication of the convergence given. That this issue had not occurred to most candidates was clear from their usual answer to part **(ii)** which was "$t \neq 2$", as opposed to the correct "$t < 2$".

The third part of the question was generally well done, with most candidates opting to differentiate twice rather than use the straightforward binomial expansion.

Answer: **(iii)** 1.5

7 Almost all candidates realised that they needed to find the expected values of the two given functions by integration and then multiply by an appropriate constant, but final answers were often poorly written. In part **(i)** the answer "$k = \frac{4}{3}X$" was common, and in part **(iii)**, $\frac{10}{9}X$ instead of $\frac{10}{9}M$.

The formula in part **(ii)** needed much more convincing derivation from many and there were numerous obvious instances of attempting to work back from the answer. Good answers showed clearly where the expression $\frac{x^3}{k^3}$ came from and then convincingly explained why this had to be cubed to give the CDF of M.

Section B: Mechanics

8 Many candidates found this an easy start, although some did not take note of the instruction to use energy and used only forces and *suvat* equations. This scored a maximum of 2 marks. Of those who correctly used energy, a pleasing number got the signs right; a few unfortunately had their calculators in radian mode.

Answer: 0.968

9 This question was much better done than the corresponding one last year. Most candidates could get the correct angle in part **(i)** and many correct answers to both parts were seen, although a few thought that the acceleration was $r\omega$ or omitted the mass. A few weak candidates attempted to resolve along the string or used tension twice in the same equation (possibly trying to mimic last year's question).

Answers: **(i)** 0.16 **(ii)** 7.91

10 The question on the new topic of moments was well done by most. Correct answers were often seen to both parts and only the weakest did not use trigonometry sensibly. It would be good if all candidates could be encouraged to annotate their equations to explain what they are; a simple "M(B)" or "R($\rightarrow$)" would help candidate and Examiner alike.

Answers: **(i)** 51.3° **(ii)** 3.57 m

11 Again, this question was largely done well. Only a few candidates were unconvincing in deriving the differential equation (attempts such as "32000 = (800a + 20v)v" gained no credit).

Most knew how to solve the differential equation and could get at least as far as finding the correct constant of integration, though many did not read the question carefully and did not attempt to make v the subject of their formula. Some did not spot that the integral was of the form $f'(x)/f(x)$ and used partial fractions; only a few thought it was a $\tanh^{-1}$ integral. A good number could find the limiting velocity.

Answers: **(ii)** $v = 40\sqrt{1 - e^{-t/20}}$; 40 ms^{-1}

12 In this question it was often very hard to see what candidates were doing. Clear "before and after" diagrams indicating the notation used are all but essential in questions on this topic. Further, although the question asks for an angle, the working is very much easier if it is done entirely in separate components; otherwise the trigonometry can get quite messy. Nevertheless, most wrote down two relevant equations in the x-direction and used the fact that the y-component of the velocity of the white ball is unchanged.

Part **(ii)** was another "show that" and again it was not often convincingly answered. It was expected that a proper use of inequalities was used, rather than merely inserting the limiting values of e into the equations.

Answer: **(i)** 85.05° to x-axis

13 Projectiles on sloping planes are a standard topic, but this question was found fairly hard. Some candidates misinterpreted the question and assumed that 40° was the angle between the initial velocity and the horizontal instead of the plane. Those who took axes parallel and perpendicular to the plane easily found that at $t = 2.5$ the value of y was 17.4, but many thought that this was the answer and did not realise that to find the vertical distance this had to be divided by cos 10°.

Many got part **(ii)** right but there were several attempts to find the time to maximum height and double it, which does not work on a sloping plane. Again, many found the correct value of t but were then unable to turn that into the correct distance along the slope, often omitting the component parallel to the plane of acceleration due to gravity. Only one or two candidates used the trajectory equation.

Answers: **(i)** 17.7 m **(ii)** 76.7 m

14 The scenario here is quite standard but many candidates seemed unfamiliar with it. Almost everyone got $e = 0.5$ in part **(i)** but correct answers to part **(ii)** were more rare. Many forgot to use the extra 0.5. Some went through the unnecessary process of finding the speed at the point when the string first becomes stretched. Those who did solve a quadratic equation almost always selected the correct solution.

In part **(iii)** the expression “equation of motion” seemed unfamiliar. The best candidates had no difficulty here but many had no idea what to do or did not take into account the equilibrium extension. In any SHM question, obtaining the equation in the form $\ddot{x} = -\omega^2 x$ should be done almost as a matter of course.

Again, in part **(iv)**, many more candidates needed to realise that the time for which the string was stretched was neither a whole nor a half cycle but in-between. Better candidates drew helpful diagrams to explain their method; otherwise, it is hard to give credit for apparently random inverse trigonometrical expressions leading to incorrect answers.

Answers: **(i)** 0.5 m **(ii)** 1.37 m **(iii)** $\ddot{x} = -20x$ **(iv)** 0.978 s