Cambridge International Examinations
Cambridge Pre-U Certificate

FURTHER MATHEMATICS **9795/01**

Paper 1 Further Pure Mathematics **May/June 2016**

MARK SCHEME

Maximum Mark: 120

Published

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Question	Answer	Marks	Notes
1	$\sum_{r=1}^{n}\left(8r^3+r\right)\equiv 8\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r$	M1	Splitting into separate series
	$\equiv 8\times\frac{1}{4}n^2(n+1)^2+\frac{1}{2}n(n+1)$	M1 M1	Both used good factorisation attempt
	$\equiv \frac{1}{2}n(n+1)\left\{4n^2+4n+1\right\}$		
	$\equiv \frac{1}{2}n(n+1)(2n+1)^2$	A1 [4]	Legitimate **(AG)**
2	$\begin{pmatrix}6\\2\\5\end{pmatrix}\times\begin{pmatrix}-6\\1\\4\end{pmatrix}=3\begin{pmatrix}1\\-18\\6\end{pmatrix}$	M1 A1	Attempt at vector products of the d.v.s (any suitable multiple)
	Shortest Distance = $\lvert(\mathbf{b}-\mathbf{a})\bullet\hat{\mathbf{n}}\rvert$	M1	
	$=\frac{1}{19}\begin{pmatrix}10\\-2\\5\end{pmatrix}\bullet\begin{pmatrix}1\\-18\\6\end{pmatrix}=\frac{1}{19}\left(10+36+30\right)$	B1 B1	$\lvert\hat{\mathbf{n}}\rvert$ correct Sc. Prod. **ft** correct
	$=4$	A1 [6]	
	Alternative method: **M1 A1** for common normal $\mathbf{i}-18\mathbf{j}+6\mathbf{k}$ **M1 A1** for parallel planes $x-18y+6z=-55$ and -131 **M1 A1** for Sh.D formula, $\frac{\lvert 131-55\rvert}{\lvert\mathbf{n}\rvert}=\frac{76}{19}=4$		
3 (i)	$\frac{2x^2-x-1}{2x-3}=k\Rightarrow 2x^2-(2k+1)x+(3k-1)=0$	B1	**(AG)** Shown legitimately
	For non-real x, $(2k+1)^2-8(3k-1)<0$	M1	Considering discriminant (or equivalent)
	$4k^2-20k+9<0\Rightarrow(2k-1)(2k-9)<0$	M1	Solving from $\Delta<0$
	$\Rightarrow$ no curve for $\frac{1}{2}<k=y<\frac{9}{2}$	A1 [4]	**(AG)** Must be satisfactorily explained
(ii)	TPs at $y=\frac{1}{2}$ $\quad y=\frac{9}{2}$	M1	First y (k) substituted back
	i.e. $2x^2-2x+\frac{1}{2}=0$ $\quad 2x^2-10x+\frac{25}{2}=0$	M1	Second y (k) substituted back
	$x=\frac{1}{2}$ $\quad x=\frac{5}{2}$	A1A1 [4]	
	<u>Alternative method:</u> when $\Delta=0$, **M1** x = "$-\frac{b}{2a}$" = $\frac{2k+1}{4}$ **M1** $\Rightarrow x=\frac{1}{2}$ ($y=\frac{1}{2}$) & $x=\frac{5}{2}$ ($y=\frac{9}{2}$) **A1 A1** **Note:** For finding TPs via $\frac{dy}{dx}=0$, max. **M1 A1** since qn. asks for a "deduce" method		

Question	Answer	Marks	Notes
4 (i)	Attempt at det(**M**) Det = 0 *Shown*	M1 A1 [2]	 (Or via full alternative algebraic method)
(ii)	$-x+3y+z=1$ $5x-y+2z=16$ $-x+y \quad =-2$ parametrisation attempt (or equivalent) started: e.g. set $x=\lambda$, then $y=\lambda-2$ complete attempt: $z=1+\lambda-3\lambda+6=7-2\lambda$ all correct (p.v. and d.v.) … may be in vector line eqn. form: $\mathbf{r}=\begin{pmatrix}0\\-2\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-2\end{pmatrix}$ <u>Alternative method 1:</u> **B1** as above, followed by (e.g.): Finding two distinct points on the solution line; e.g. (2, 0, 3), (0, –2, 7) **M1 A1** Then eqn. of line containing these 2 points **M1 A1** possibly **ft** for line (of intersection) of 3 planes (given by the 3 eqns.) **B1** <u>Alternative method 2:</u> **B1** as above, followed by: Vector product of any two plane normals **M1A1** Finding coords. or p.v. of any pt. on line **B1** Eqn. of line using these results appropriately **B1** for line (of intersection) of 3 planes (given by the 3 eqns.) **B1**	 B1 M1 M1 A1A1 [6]	 for all three
5	Aux. Eqn. $m^2-4m+5=0$ $m=2\pm\mathrm{i}$ Comp. Fn. is $y_C=\mathrm{e}^{2x}\left(A\cos x+B\sin x\right)$ For Part. Intgl. try $y=y_p=a\,\mathrm{e}^{2x}$ Both $y'=2a\,\mathrm{e}^{2x}$ and $y''=4a\,\mathrm{e}^{2x}$ Subst^{g}. into given d.e. & solving to find a : $y_p=24\mathrm{e}^{2x}$ Gen. Soln. $y=\mathrm{e}^{2x}\left(A\cos x+B\sin x+24\right)$	M1 A1 B1ft B1 B1 M1 A1 B1ft [8]	Including solving attempt $\left(4a-8a+5a\right)\mathrm{e}^{2x}=24\mathrm{e}^{2x}$ y_C+y_P provided y_C has 2 arbitrary constants and y_P has none. Also, A, B must be real here

Question	Answer	Marks	Notes
6 (i)	For f(x) = sinh x + sin x – 3x, f(2.5) = – 0.851… < 0 and f(3) = 1.159… > 0 Change-of-sign (for a continuous fn.) $\Rightarrow$ $2.5 < \alpha < 3$	M1 A1 [2]	or LHS < RHS and then LHS > RHS All correctly shown/explained
(ii)	$\sinh x + \sin x = \left(x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \frac{x^7}{7\,!} + \frac{x^9}{9\,!} + ...\right) + \left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^7}{7\,!} + \frac{x^9}{9\,!} - ...\right)$ $= 2x + \frac{x^5}{60} + ...$ $2x + \frac{x^5}{60} = 3x \Rightarrow (x \neq 0)\ x^4 = 60$ $\Rightarrow \alpha \approx \sqrt[4]{60}$ (2.783 158 …)	M1 A1 B1 [3]	for use of both series (attempted) **(AG)** shown legitimately
(iii)	Using $2x + \frac{x^5}{60} + \frac{x^9}{181\,440} = 3x$ with $x \neq 0$ Solving as a quadratic in x^4 $\alpha \approx 2.769\,8$ (to 4 d.p.) [c.f. actual root 2.769 7 to 4 d.p.]	M1 M1 A1 [3]	 $x^8 + 3024x^4 - 181\,440 = 0$ from $x^4 = \sqrt{2\,467\,584} - 1512$, $x = \sqrt[4]{58.854\,5...}$
7 (i)	$\lvert z^3 \rvert = 2\sqrt{2}$ $\arg(z^3) = \frac{1}{4}\pi$ $\Rightarrow z = \left(\sqrt{2},\ \frac{1}{12}\pi\right)$ cube-rooting modulus; arg ÷3 Other two roots: $\left(\sqrt{2},\ \frac{3}{4}\pi\right)$ and $\left(\sqrt{2},\ \frac{17}{12}\pi\right)$	B1B1 M1M1 A1A1 [6]	 (in at least the first case)
(ii)	Equilateral Δ with vertices in approx. correct places Area = $3 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \sin\left(\frac{2}{3}\pi\right) = \frac{3}{2}\sqrt{3}$ Accept **awrt** 2.60 (3 s.f.) from correct working	B1 M1A1 [3]	 Give M1 for any correct area

Question	Answer	Marks	Notes
8 (i) (a)	*(table G below)*	M1 A1 [2]	for mostly correct for all correct
(b)	$(S, \times_{63})$ closed, since no new elements in table $\times_{63}$ is associative (given) 1 is the identity element Each (non-identity) element has a unique inverse: $2 \leftrightarrow 32$, $4 \leftrightarrow 16$ and 8 is self-inverse	B1 B1 B1 [3]	All must be identified
(ii) (a)	*(table H below)*	B1 B1 B1 B1 [4]	for last 3 elements (any forms) for identity row/column (green) for easy elements (gold) or $\geqslant$ 14 others for all
(b)	Proper subgroups of H are (condone inclusion of $\{e\}$ and H): $\{e, x\}$, $\{e, xy\}$, $\{e, yx\}$ and $\{e, y, y^2\}$	B1B1 [2]	B1 Any 2; +B1 all 4 and no extras
(c)	G and H are NOT isomorphic e.g. Different numbers of self-inverse elements / elements of order 3 or G cyclic, H non-cyclic or G abelian, H non-abelian	B1 [1]	Correct conclusion WITH a valid reason

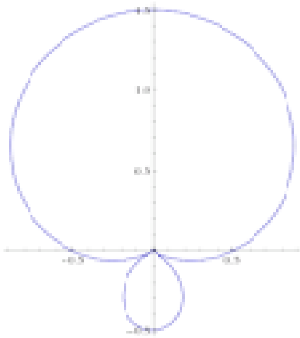
Table for 8 (i) (a):

G	**1**	**2**	**4**	**8**	**16**	**32**
1	1	2	4	8	16	32
2	2	4	8	16	32	1
4	4	8	16	32	1	2
8	8	16	32	1	2	4
16	16	32	1	2	4	8
32	32	1	2	4	8	16

Table for 8 (ii) (a):

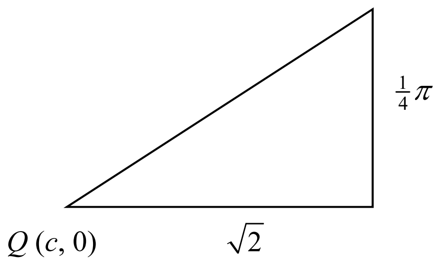
H	e	x	y	y^2	xy	yx
e	e	x	y	y^2	xy	yx
x	x	e	xy	yx	y	y^2
y	y	yx	y^2	e	x	xy
y^2	y^2	xy	e	y	yx	x
xy	xy	y^2	yx	x	e	y
yx	yx	y	x	xy	y^2	e

Question		Answer	Marks	Notes
9	(i)	$\alpha+\beta+\gamma=a$, $\alpha\beta+\beta\gamma+\gamma\alpha=b$ and $\alpha\beta\gamma=c$	B1B1 [2]	B1 any 2 correct; + B1 all 3 correct
	(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	M1	
		$=a^2-2b$	A1	
		$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$	M1	
		$=b^2-2ac$	A1 [4]	
	(iii)	$(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)$		
		$=\left(\alpha\beta-2\beta^2\gamma-2\alpha^2\gamma+4\gamma^2\alpha\beta\right)(\gamma-2\alpha\beta)$	M1	
		$=\alpha\beta\gamma-2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2) + 4\alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)-8(\alpha\beta\gamma)^2$	M1	Collecting up in terms of the symmetric fns.
		$=c-2\left(b^2-2ac\right)+4c\left(a^2-2b\right)-8c^2$	M1	Use of **(i)**'s and **(ii)**'s results
		$=c\left(1+4a+4a^2\right)-2\left(b^2+4bc+4c^2\right)$		
		$=c(2a+1)^2-2(b+2c)^2$	A1 [4]	**legitimately**
		Alternative method:		
		Using $\alpha\beta\gamma=c$, $(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)$		
		$=\left(\alpha-\frac{2c}{\alpha}\right)\left(\beta-\frac{2c}{\beta}\right)\left(\gamma-\frac{2c}{\gamma}\right)$		
		$=\frac{1}{\alpha\beta\gamma}\left(\alpha^2-2c\right)\left(\beta^2-2c\right)\left(\gamma^2-2c\right)=$		
		$\frac{1}{c}\left((\alpha\beta\gamma)^2-2c\sum\alpha^2\beta^2+4c^2\sum\alpha^2-8c^3\right)$		
		$=\frac{1}{c}\left(c^2-2c\left[b^2-2ac\right]+4c^2\left[a^2-2b\right]-8c^3\right)$		
		= etc. as above		
	(iv)	One root is the product of the other two $\Leftrightarrow$ $(\alpha-2\beta\gamma)(\beta-2\gamma\alpha)(\gamma-2\alpha\beta)=0$		
		$\Leftrightarrow$ $c(2a+1)^2=2(b+2c)^2$	B1	**legitimately**
		Must reason $\Rightarrow$ and $\Leftarrow$ explicitly (or together)	[1]	

Question	Answer	Marks	Notes
10 (i)		M1A1	$\frac{1}{2}+\sin\theta=0$ when $\theta = \frac{7}{6}\pi$, $\frac{11}{6}\pi$
		B1	Symmetry in y-axis
		B1	$\left(\frac{1}{2},\ 0\right)$ on initial line
		B1	Correct upper portion
		B1 [6]	Correct lower portion
(ii)	$A=\left(\frac{1}{2}\right)\int_0^{2\pi}\left(\frac{1}{2}+\sin\theta\right)^2\,\mathrm{d}\theta$	M1	Penalise incorrect multiples with final A0
	$=\frac{1}{2}\int_0^{2\pi}\left(\frac{1}{4}+\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$	M1	Double-angle formula
	$=\frac{1}{2}\left[\frac{3}{4}\theta-\cos\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$	A1	correctly integrated 3 suitable terms
	$=\frac{3}{4}\pi$	A1 [4]	

Question	Answer	Marks	Notes
11 (i)	$F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8$	B1 [1]	all
(ii) (a)	$p_2(x) = 1 + \dfrac{1}{x+1} = \dfrac{x+2}{x+1}$	B1	
	$p_3(x) = \dfrac{2x+3}{x+2}$	B1	
	$p_4(x) = \dfrac{3x+5}{2x+3}$	B1 [3]	**(AG))**
(b)	$p_n(x) = \dfrac{F_n\ x + F_{n+1}}{F_{n-1}\ x + F_n}$	B1	
	Result is true for $n = 2$ (and 3 and 4)	B1	May be mentioned in later in their "round up"
	Assuming $p_k(x) = \dfrac{F_k\ x + F_{k+1}}{F_{k-1}\ x + F_k}$ (not separate from their conjecture)		
	$p_{k+1}(x) = 1 + \dfrac{F_{k-1}\ x + F_k}{F_k\ x + F_{k+1}}$	M1	
	$= \dfrac{F_k\ x + F_{k+1}}{F_k\ x + F_{k+1}} + \dfrac{F_{k-1}\ x + F_k}{F_k\ x + F_{k+1}}$		
	$= \dfrac{\left(F_k + F_{k-1}\right)\ x + \left(F_k + F_{k+1}\right)}{F_k\ x + F_{k+1}}$	M1	Collecting coeffts. into successive Fib. terms
	$= \dfrac{F_{k+1}\ x + F_{k+2}}{F_k\ x + F_{k+1}}$	A1	
	which is the required formula with $n = k + 1$. Accept this as sufficient that proof follows by induction.	[5]	

Question	Answer	Marks	Notes
12 (i)	$y = \ln\left(\tanh \tfrac{1}{2}x\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\tanh \frac{1}{2}x} \cdot \tfrac{1}{2}\operatorname{sech}^2 \tfrac{1}{2}x$	M1A1	
	$= \operatorname{cosech} x$	A1 [3]	**(AG)**
(ii) (a)	$L_n = \displaystyle\int_n^{2n} \sqrt{1 + \operatorname{cosech}^2 x}\, dx$	M1	
	$= \displaystyle\int_n^{2n} \coth x\, dx$	A1	
	$= \left[\ln(\sinh x)\right]$	A1	correct integrn.
	$\ln\left(\dfrac{\sinh 2n}{\sinh n}\right) = \ln\left(\dfrac{e^{2n} - e^{-2n}}{e^n - e^{-n}}\right)$	M1	
	$\approx \ln\left(\dfrac{e^{2n}}{e^n}\right)$, for large n, $= \ln\left(e^n\right) = n$	A1 [5]	**legitimately**
	OR $\ln\left(\dfrac{\sinh 2n}{\sinh n}\right) = \ln\left(2\cosh n\right) = \ln\left(e^n + e^{-n}\right)$ **M1**		
	$\approx \ln\left(e^n\right)$ for large n, $= n$ **A1**		**legitimately**
(b)	Method (sketch or statement) to indicate that C asymptotically "merges" with the x-axis	M1	
	so that C is approximately a horizontal straight-line from $(n, 0)$ to $(2n, 0)$	A1 [2]	
13 (i) (a)	Let $y = \sec^{-1}x$, i.e. $\sec y = x$		
	$\Rightarrow \cos y = \dfrac{1}{x} \Rightarrow y = \cos^{-1}\left(\dfrac{1}{x}\right)$	B1	
	Then $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{d}{dx}\left(\cos^{-1}\dfrac{1}{x}\right)$		
	$= -\dfrac{1}{\sqrt{1-(1/x)^2}} \times \dfrac{-1}{x^2}$	M1	(Using MF20 and the *Chain Rule*)
	$= \dfrac{1}{x\sqrt{x^2-1}}$	A1 [3]	**(AG)**
	[Allow **M1 A1** for valid non-"deduced" approaches]		

Question	Answer	Marks	Notes
(b)	$\int \sec^{-1} x \,.\, 1 \;\mathrm{d}x$	M1	Use of integration by "parts"
	$= x \,.\, \sec^{-1} x - \int x \,.\, \dfrac{1}{x\sqrt{x^2-1}} \,\mathrm{d}x$	A1 A1	
	$= \left[x \,.\, \sec^{-1} x - \cosh^{-1} x \right]$	A1 [4]	Condone lack of "+ C"
(ii) (a)	$\dfrac{1}{x\sqrt{x^2-1}} = \dfrac{1}{\sqrt{2}} \;\Rightarrow\; x^2\left(x^2-1\right) = 2$		
	$\Rightarrow\; x^4 - x^2 - 2 = \left(x^2-2\right)\left(x^2+1\right) = 0$	M1	
	$\Rightarrow\; x = \sqrt{2}$ and $y = \frac{1}{4}\pi$	A1 A1	i.e. $P = (\sqrt{2}, \frac{1}{4}\pi)$
			
	$\dfrac{\frac{1}{4}\pi}{\sqrt{2}-c} = \dfrac{1}{\sqrt{2}}$	M1 A1	or by $y - \frac{1}{4}\pi = \dfrac{1}{\sqrt{2}}\left(x - \sqrt{2}\right)$ & $y = 0$
	$c = \sqrt{2} - \dfrac{\pi\sqrt{2}}{4}$	A1 [6]	i.e. $Q = \left(\sqrt{2} - \dfrac{\pi\sqrt{2}}{4},\; 0\right)$
(b)	Area $\Delta = \dfrac{1}{2} \times \dfrac{\pi\sqrt{2}}{4} \times \dfrac{\pi}{4} = \dfrac{\pi^2\sqrt{2}}{32}$	B1	
	Area under curve $= \sqrt{2} \,.\, \dfrac{\pi}{4} - \ln\left(1+\sqrt{2}\right)$	B1	using **(iii)**'s answer and the limits $(1, \sqrt{2})$
	Then $R = \dfrac{\pi^2\sqrt{2}}{32} - \dfrac{\pi\sqrt{2}}{4} + \ln\left(1+\sqrt{2}\right)$	M1	Difference in areas
	$= \ln\left(1+\sqrt{2}\right) - \dfrac{\pi(8-\pi)\sqrt{2}}{32}$	A1 [4]	**(AG)**