FURTHER MATHEMATICS

Paper 9795/01
Further Pure Mathematics

Key messages

There are significant points that candidates need to bear in mind before they sit this paper, many of which are highlighted again in the later comments on individual questions. Firstly, this is a long and tough paper and few candidates complete attempts at all questions. Thus, for each individual candidate, it is very important to choose a suitable order in which to attempt the questions, one that maximises the opportunities to gain high marks. Also, there is consequently little time in which candidates will have an opportunity to review or correct working, so care must be taken to ensure that work is done without the need for re-evaluation. Oftentimes, candidates are best served by a thoughtful approach to each question that considers – in particular – how the question is structured (where appropriate); many candidates frequently ignore the very careful signposts offered within the question. In a similar vein, candidates must be aware of how appropriate the working they are writing down is to the demand of that question or question-part; producing up to a page-and-a-half of working for a result that has been assigned just 1 or 2 marks is clearly inappropriate, and candidates should allow themselves to be guided by the number of marks.

Another key point is that many candidates seem to lack confidence when deploying prior learning in the further maths setting – almost as if they have forgotten about them altogether – and they should be aware that understanding and fluency in GCSE-level and (single) Maths-9794-level techniques are taken as 'background/assumed knowledge' within the further mathematics papers. Moreover, when such work is required within one of the further pure mathematics questions, they are often assigned relatively few marks compared to those assigned to the higher level topics of work that are primarily being tested here.

Finally, a consistent shortfall in the scoring of marks arises whenever an explanation, justification or proof is required of candidates: or, in a similar vein, a diagram with all appropriate points/lines/asymptotes, etc., clearly marked on it. Such opportunities arose this year in **Questions 3, 6, 9, 10, 11** (especially) and **12** and candidates were not averse to skipping through them without taking sufficient care with the 'small details'.

General comments

Overall, the quality of candidates' work continues to be of a very high standard and, despite the very off-putting appearance of the final question this year, almost all candidates made some attempt at every question on the paper. Marks for the first nine questions were very high indeed, falling (on average) only 1 or 2 short of the maximum mark available. The work produced within scripts for these questions revealed a cohort of capable and well-prepared candidates.

It was **Questions 10–12** that provided the real challenge to candidates, with a very long and tough **Question 12** the primary culprit in this respect: this probably accounts for the absence of anyone scoring full marks this year.

Comments on specific questions

Question 1

This was a gentle introduction to the paper and was well done by almost all candidates. The only slips of any consequence related to carelessness with minus signs and in forgetting to make sure that the values for a and b were paired correctly.

Answer: $a = \pm 5$, $b = \mp 2$

Question 2

This question was also found to be quick and easy by almost all candidates. Most took the expected route via $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$, although a small number opted to find the equation with roots α^2, β^2 and γ^2 instead and then pick out the coefficient of the quadratic term. Most comments correctly spotted that (at least) one of the roots had to be complex in order for a sum of squares to be negative, although rather too many candidates seemed to be unaware of the difference between the terms 'complex' and 'imaginary' – they were not punished for any lack of clarity in this respect, however. Many candidates went further still and explained that exactly one root was real and the other two were complex conjugates; though this, of course, follows from the fact that the original equation has all of its coefficients real.

Answer: -2

Question 3

This question was also well-handled by almost everyone, although quite a few candidates overlooked the need to show some sort of scale on their sketch-graph (the point (1, 0) would have sufficed). Apart from a small number of minor mistakes in the integration work, **part (ii)** went very smoothly too.

Answer: **(ii)** $\dfrac{\pi}{1+2\pi}$

Question 4

For the most part, this was all done very capably by almost all candidates. The few exceptions seemed to be those who either couldn't recall the correct formula or who had difficulty interpreting the result that appears in the examination's accompanying Formula Booklet. Following that, the key is to realise that $1 + \left(\dfrac{dy}{dx}\right)^2$ needs to be a perfect square. Those who managed to clear these hurdles gained 6 or 7 of the marks; those who did not got 1 or 2 at best.

Answer: 96π

Question 5

This question was also done very well overall. **Part (i)** was intended to be a standard piece of bookwork, requiring candidates to turn the statement $y = \tanh^{-1}x$ into $\tanh y = x$ and use the given definition to deduce the required result. However, some preferred to simply verify by substitution instead and this works equally well. **Part (ii)** was open to a variety of approaches – the mark scheme for the paper has all three of those that were seen, with the first two appearing for the most part and with approximately equal frequency.

Answer: **(ii)** $\frac{1}{4}\ln 3$

Question 6

Questions on rational functions and their graphs have proved to be immensely popular with candidates over recent years and this year's question proved no exception. However, there were still a few hiccups to be found on the scripts, the principal ones involving oversights or miscalculations regarding the horizontal asymptote, errors in using the *Quotient Rule of Differentiation* in **part (ii)** and incorrectly drawn graphs in **part (iii)**, where the left-hand 'half' was occasionally missing altogether or, more commonly, drawn below the x-axis.

Answers: **(i)** $x = -1$, $y = 1$ **(ii)** $\dfrac{dy}{dx} = \dfrac{2(x-1)}{(x+1)^3}$ $(1, \frac{1}{2})$

Question 7

This was a relatively routine second-order differential equation question and it was dealt with appropriately by almost everyone. Just occasionally, a candidate would overlook the need to produce an answer *without* complex coefficients: it is expected that a complementary function of the form $Ae^{2ix} + Be^{-2ix}$ will be turned into one of the form $C\cos 2x + D\sin 2x$ in order to gain full marks.

Answers: **(i)** $k = 2$ **(ii)** $y = \cos 2x + (2x + \frac{1}{2})\sin 2x$

Question 8

This vectors question was handled very well and with an interesting variety of approaches. Almost everyone ignored the straightforward trigonometric approach in **part (i)(b)**, for instance, with the most usual alternative involving the well-known formula quoted in the Formula Booklet. In **part (i)(a)**, a very small number of candidates confused the angle between **d** and **n** with the angle between the line and the plane; otherwise, things progressed nicely, apart from those candidates who worked out the distance between the two planes in **part (iii)** in bits and muddled up some scale-factor or other along the way.

This is a good point at which to mention another unhelpful feature of many candidates' working, where the tendency is simply to write down a whole load of numbers with little indication as to what they represent; this may be easy to interpret positively by the marker when the right answer appears at the end of it all, but it is often of little help otherwise when there is no apparent structure or process to what has been committed to writing. Candidates need to be aware that an examiner is unable to reward what they think the candidate might be doing, and method marks require some visible structure that can be followed.

Answers: **(i)(a)** $\frac{20}{21}$ **(b)** $\frac{7}{4}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ **(c)** 5 **(ii)** $\frac{13}{2}$

Question 9

This question on a mixed bag of tricky calculus matters, involving both inverse trigonometric functions and hyperbolic functions, was expected to be found very tough, since little like it has appeared on the papers before now. However, it was handled almost universally successfully by candidates. The only possible point of interest arose with the taking of square-roots and the need to justify that only the positive one applied in each of **parts (i)** and **(ii.**

Answer : **(iii)** $\frac{-\sec^{-1}x}{x} + \frac{\sqrt{x^2-1}}{x}$ $(+\ C)$

Question 10

This was the point at which things got significantly more demanding on this paper. Part of the reason for the extra difficulty on this question is, firstly, that it is almost entirely algebraic. Secondly, a numerical or sign error early on naturally led to the loss of several of the following marks. Finally, **part (iv)** required a little bit of constructive thought from candidates and the time constraint factor, for the paper as a whole, was probably starting to tell.

To begin with, there were quite a few candidates who did not appreciate the standard 'partial fraction form' that was being demanded of them; these few candidates were thus barred from any chance of successful progress with either of **parts (i)** or **(ii)**. Candidates did not deal with the partial fractions using the '*Cover-Up Method*', which would have meant that the terms in **part (i)** could simply have been written straight down. Beyond that small detail, however, it was noticeable that most candidates preferred to work with

$\frac{1}{2(k-1)} - \frac{1}{k} + \frac{1}{2(k+1)}$ rather than $\frac{\frac{1}{2}}{k-1} - \frac{1}{k} + \frac{\frac{1}{2}}{k+1}$, and keeping the $\frac{1}{2}$s well away from the denominators makes the working much less awkward. The other helpful hint (for future reference) would be that it is easier to cancel terms in **part (ii)** when the two positive terms appear side-by-side *before* attempting to subtract the remaining one; in this way, the reciprocals of the integers in the summation generally appear once as a positive and once as a negative and the cancelling of terms is then far more readily identified (and justified).

Part (iv) was usually rather poorly handled, as most candidates were unable able to figure out where the $\frac{27}{24}$ and the $\frac{29}{24}$ had to come from.

Answer : **(i)** $\dfrac{\frac{1}{2}}{k-1} - \dfrac{1}{k} + \dfrac{\frac{1}{2}}{k+1}$

Question 11

Groups questions have traditionally been found difficult in all cases except the most straightforward ones. This one was a mixture of the easy and the tough and, although almost all candidates scored some marks, they still managed to miss out on several more. While **parts (i)(a)**, **(b)** and **(iii)(a)**, **(b)** provided easy access to 6-8 marks for most, parts **(ii)** and **(iii)(c)** were sources of considerable misunderstanding as to the true nature of the task.

The Principal Examiner would direct future candidates and their Teachers to the mark-scheme for **part (ii)** for guidance, as there were lots of marks unacquired by those who undoubtedly thought they had scored most, if not all, of the 5 available to them. The real issue here was that the elements of *G* were not just 2×2 matrices, but rather those whose determinants were equal to 1. It was, therefore, insufficient (for example) merely to point out that the inverse of a given matrix existed (which is a known result for all such matrices with non-zero determinant); it was important to justify that it was actually **in** *G* because its determinant was 1. A similar statement was also required to justify the assertion that the 2×2 identity matrix **I** was in *G*. The whole point of **part (i)** was to supply candidates with the means to justify the '*closure*' property, though this fact was overlooked by rather a lot of them.

By the time it came to **part (iii)(c)**, most candidates seemed to have decided it was time to move on to the final question as there were very few serious suggestions offered here. Even among this minority, most did not notice the requirement that the second subgroup needed to be in *G* also and they merely proposed another cyclic group of order 6. The anticipated subgroup was that of the rotations through multiples of 60° but there were alternatives involving a variation on the theme of **K**; however, as these were about fifty-fifty for correctness, it was not entirely clear whether some candidates were just luckier guessers than their peers. Even so, it was not apparent to all but a few that they should make any attempt to justify that their proposal was indeed both a subgroup of *G* **and** isomorphic to *H*.

Answers: **(i)(a)** $\mathbf{AB} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ det $\mathbf{A} = ad - bc$, det $\mathbf{B} = eh - fg$ **(iii)(b)** $n = 6$

Question 12

This was the most demanding question on the paper. However, for any candidate prepared to hold their nerve there were plenty of carefully directed marks available, and it was only the induction in **part (iii)** that was actually especially tough; even then, there are always 2 or 3 marks available on any induction question for starting off the process.

Oddly enough, one of the biggest hurdles to steady progress lay with a lot of candidates' refusal to adopt the 'c' for $\cos\theta$, 's' for $\sin\theta$ abbreviation suggested at the outset. The difficulties that followed revolved around the extraordinary amount of extra writing that these candidates had to undertake and the difficulty that naturally arose from trying to sort out the sea of letters that they were then confronted with. The next factor that largely went unnoticed is that key parts of **parts (i)**, **(ii)(b)** *and* **(iii)** all fall nicely out when using the most basic of trigonometric identities, namely $c^2 + s^2 = 1$. Fortunately, all parts of this question could be successfully tackled by any one of several methods, and with the necessary contributory bits applied in a variety of orders.

FURTHER MATHEMATICS

Paper 9795/02
Further Applications of Mathematics

Key messages

- Give full details in 'show that' questions
- Write clearly, concisely and precisely when asked to 'explain'
- Understanding of concepts of modelling assumptions and conditions, etc., especially in statistics
- Ensure that the answer given matches the question set

General comments

This paper contained a couple of tricky question parts which most candidates found very difficult. Generally, it was still evident that many candidates were not well enough versed in some basic definitions or methodologies. However, most candidates were good at applying standard techniques; it was usually when deeper understanding was tested that misconceptions and gaps were revealed.

Many questions, especially the longer, less structured ones, required candidates to carry out a detailed and complex chain of reasoning and calculation; the more successful candidates tended to be those who could do this in a logical and methodological manner; use of sensible notation also helped greatly. Candidates should also be encouraged to include indicators to help the flow of reasoning; for example, the use of logical connectors, the naming and quoting of standard results and making it clear where previous results are being utilised can easily improve and clarify an otherwise apparently unstructured mass of work.

Candidates should appreciate that if the answer is given in a particular question then the onus is on them to demonstrate that they have reached this answer properly and not simply 'fudged it'.

Candidates should be encouraged to take a consistent approach with their accuracy, working with either the full answers on their calculators or, at least, several more significant figures than are to be quoted in their final answers. Candidates should also understand that if an answer is required to 3 significant figures, say, then an answer of 0.2 is insufficient.

Candidates should also be encouraged to strike out any work which they do not wish to be assessed and to do this as neatly and clearly as possible.

Comments on specific questions

Section A: Probability

Question 1

Candidates should be encouraged to learn definitions precisely; it was evident from the wide range and style of different answers to **part (i)** that candidates have an intuitive, rather than precise, understanding of what is meant by 'confidence interval'.

In **part (ii)** while most candidates understood in broad terms what was required, many candidates were clearly unfamiliar with how to deal with a small sample.

Answers: **(ii)** (35.6, 43.8)

Question 2

Most candidates found **part (i)** straightforward. However, when it came to applying the probability generating function it became clear that the concept itself was not so well understood and many candidates used a different method to obtain the answers in **part (ii)**; since the question specified the required method this did not achieve full marks.

Answers: **(i)** $\frac{1}{3}t^{-1}+\frac{2}{3}t^{2}$ **(ii)(a)** 10 **(b)** $\dfrac{4480}{19683}$

Question 3

A large number of candidates found **part (i)** of this question straightforward and achieved full marks; it is simply a question of recalling and applying a standard result.

A high number of candidates made basic algebraic or arithmetic errors in **part (ii)** which caused the unnecessary loss of marks; candidates should be encouraged to check their working carefully.

Answers: **(i)** (0.200, 0.440) **(ii)** 3610

Question 4

Part (i) was very standard and most candidates managed it without difficulty. However, candidates should be encouraged to set work out logically; for example, here they should have started from the general rule $\int_{-\infty}^{\infty} \mathrm{f}(x)\mathrm{d}x=1$ and then shown, with the correct use of limits, how this applied to the particular question. This approach helps partial credit to be awarded.

In **part (ii)**, while a few candidates appeared completely confused by the idea of a cumulative distribution function (CDF) most realised that the given function had to be integrated piecewise. The most common mistake by far was in not understanding that constants of integration were required and that these had to ensure continuity of the CDF. Some lost a mark unnecessarily because they did not appreciate that a CDF must tend to 1 as x tends to infinity, simply including a ‘0 otherwise’ piece intending to account for all of the ‘uninteresting’ portion of the function.

In **part (iii)** many candidates made the question harder than it needed to be by actually finding, or attempting to find, the value of the upper quartile; all that was required was to find where 0.25 lies in the distribution, a fact that can be readily determined from the CDF.

Answers: **(i)** $\dfrac{3}{5}$ **(ii)** $\begin{cases} \frac{3}{5}(x+1) & -1\leqslant x<0 \\ \frac{3}{5}(x-\dfrac{x^3}{3}+1) & 0\leqslant x\leqslant 1 \\ 0 & x<-1 \\ 1 & x>1 \end{cases}$ **(iii)** greater

Question 5

Most candidates found this question entirely straightforward.

Answers: **(i)** 0.111 **(ii)** 92 seconds **(iii)** 0.761

Question 6

The majority of candidates could derive the given MGF for **part (i)**. A small number of candidates started from the given formula in the formulae booklet but the question clearly required 'the definition of the moment generating function' as the start point.

Once again, applying the MGF in **part (ii)** proved harder and many candidates were confused as to what was required. When finding the variance candidates should be encouraged to state explicitly the value of E(X), even when this is zero.

Answer: **(ii)** $\frac{1}{3}, \frac{1}{5}$

Question 7

Most candidates found **part (i)** entirely straightforward. Unfortunately, **part (ii)** proved to be very difficult; possibly because candidates had found the distribution of the total mass in **part (i)** and they were not then alert to the fact that the total mass of the can and the mass of the container are not independent; a fact which has important consequences on the required methodology. Consequently, very few candidates obtained full marks on this question.

Answers: **(i)** 0.0325 **(ii)** 0.373

Section B: Mechanics

Question 8

Most candidates found **part (i)** straightforward.

In **part (ii)** most candidates could find the radial component but many did not seem to be able to find the transverse component correctly, often incorrectly using the answer to **part (i)** instead. Many candidates did not, in any case, add the components correctly.

Answers: **(i)** 4.2 **(ii)** 0.666

Question 9

Most candidates managed **part (i)** although many derived the formula from scratch while the question did not require this since the command words were 'Write down'.

In **part (ii)** many candidates seemed to get themselves into a tangle in their working; this is one of those questions which demands a clear and logical approach. There are many different methods for obtaining the correct answer; candidates should be encouraged to spend some time thinking about which approach is likely to be easiest and most efficient. Candidates should also be encouraged to check their working carefully, especially when their final answer does not seem likely; a dropped minus sign or missing factor should be straightforwardly recoverable with care.

Answers: **(i)** $y = x\tan\theta - \frac{x^2(1+\tan^2\theta)}{320}$ **(ii)** 63.8

Question 10

Again, this question was largely done well. Many candidates derived the value of the resistive force in **part (i)** which was not necessary. Such candidates should have made it clear where this value came from when it was subsequently used in **part (ii)**. Candidates should also be strongly encouraged to identify where equations have come from rather than simply to write them down. This is again to their benefit since it makes it far easier to award partial credit where it is deserved.

Answers: **(i)** 4500 kJ **(ii)** 3125

Question 11

Almost all candidates found **part (i)** straightforward.

In **part (ii)** most candidates knew what was required in broad terms and so high marks were common. Once again, candidates should be encouraged to indicate (concisely) the meaning of equations that they present. Most candidates could not convincingly deal with the inequality. Since the answer was given the onus was on the candidate to explain why P must be at most 6.4; this could be done either algebraically with care or by considering the limiting case and providing a clear explanation. Many candidates seemed content to simply change an = sign into a $\leqslant$ sign at the end of their working which did not suffice.

Question 12

Most candidates found this question straightforward.

Answer: **(ii)** 116

Question 13

This question proved to be very difficult. Once again, it was clear that many candidates found it hard to present the structured working that was required to find the correct answer in both parts. The most common logical error was simply to assume that because a particle has enough energy to reach the top then it necessarily will but this is in fact determined by dynamical, rather than energetic, considerations.

Many candidates appeared to get a little lost in their working; sometimes it might be a good strategy for them simply to restart the question from scratch, if time permits.

Sign errors were very common, especially when considering conservation of energy.

Answers: **(i)** 5.48 **(ii)** 43.8°

Question 14

This question proved again to be very difficult.

Many candidates assumed that both strings were taut but careful reading of the scenario showed that, in fact, at any given time at most one string was taut and for much of the time both were slack.

Hardly any candidates got the answer to **part (iii)** correct; it was in fact quite easy if the scenario was properly understood and working carefully structured. Candidates must be encouraged to read the question carefully and to draw diagrams to aid their understanding.

Answers: **(i)** $\ddot{x} = -10x$ **(ii)** 0.316 **(iii)** 4.52